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A determinant formula for the quotient of the relative class numbers of imaginary abelian number fields of relative degree 2

By

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Abstract

We give a determinant formula for the quotient of the relative class numbers of imaginary abelian number fields of relative degree 2, which is a generalization of Endô's formulas for the m th cyclotomic field, m an odd integer, and its quadratic extension.

§ 1. Introduction

Let p be an odd prime. For an integer u let $R_p(u)$ and $R'_p(u)$ be the integers such that

$$R_p(u) \equiv u \pmod{p}, \quad 0 \leq R_p(u) < p$$

and

$$R'_p(u) \equiv u \pmod{p}, \quad -\frac{p}{2} < R'_p(u) < \frac{p}{2},$$

respectively. For an integer u coprime to p , let u^{-1} be an integer with $uu^{-1} \equiv 1 \pmod{p}$. We have already obtained a lot of determinant formulas for the p th cyclotomic

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field $\mathbf{Q}(\zeta_p)$, ζ_p a primitive p th root of unity. For example,

$$(1.1) \quad \det(R_p(uv^{-1}))_{1 \leq u, v \leq (p-1)/2} = (-1)^{\frac{p-3}{2}} p^{\frac{p-3}{2}} h_p^*,$$

$$(1.2) \quad \det(R'_p(uv^{-1}))_{1 \leq u, v \leq (p-1)/2} = \begin{cases} 2^{\frac{p-1}{\text{ord}_p(2)}-1} p^{\frac{p-3}{2}} h_p^* & \text{if } 2 \mid \text{ord}_p(2), \\ 0 & \text{otherwise,} \end{cases}$$

$$(1.3) \quad \det((-1)^{R_p(uv^{-1})})_{1 \leq u, v \leq (p-1)/2} = (-1)^{\frac{p-1}{2}} \frac{2^{\frac{p-3}{2}}}{p} \prod_{\chi \in X^-} (1 - \chi(2)) \cdot h_p^*,$$

where $\text{ord}_p(2)$ is the order of 2 modulo p , X^- the set of odd characters of the field $\mathbf{Q}(\zeta_p)$ and h_p^* the relative class number of the field $\mathbf{Q}(\zeta_p)$.

The determinant in the formula (1.1) is called Maillet determinant (See [1]) and the one in (1.3) could be called Dem'janenko determinant. The formulas (1.1) and (1.3) are special ones of the generalized formulas in [3], [14], [17] and [18]; and the formula (1.2) a special one of [3], [15] and [17]. Funakura [7] gave, up to sign, a generalized formula of (1.3) for the m th cyclotomic field, m an odd integer.

As a corresponding formula to (1.3), we shall obtain by the formula (2.5) in Corollary 2.3

$$(1.4) \quad \det((-1)^{R'_p(uv^{-1})})_{1 \leq u, v \leq (p-1)/2} = \begin{cases} -2^{\frac{p-3}{2}} \frac{h_{4p}^*}{h_p^*} & \text{if } p \equiv 3 \pmod{4}, \\ 0 & \text{otherwise,} \end{cases}$$

where h_{4p}^* is the relative class number of the $4p$ th cyclotomic field. Kanemitsu and Kuzumaki [15, Corollary 4] have already obtained the formula (1.4), up to sign, under some condition.

The aim of this paper is to give a determinant formula for the quotient of the relative class numbers of imaginary abelian number fields with relative degree 2, which is a generalization not only of the formula (1.4) but also of Endô's formulas in [2] and [4]. As does the formula in [14], our determinant formula has a parameter b . By taking $b = fm + 1$ (fm : the conductor of "the larger field") we obtain the formula in [2, Theorem 1 for $k = 1$]; by taking $b = 2$, the one in [4, Theorem 1]; by taking $b = g$ (g : a primitive root modulo p), the one in Corollary 2.5, in which the elements of determinant are coefficients of some digit expression as in [11].

Our result would be an answer to the inquiry of Kanemitsu and Kuzumaki [15, p.285] about the relation between Tsumura's and the author's generalized Dem'janenko determinants and (generalization of) Endô's determinants S_p , T_p and U_p in [5]. The determinants S_p , T_p and U_p are special ones of the left-hand sides of (2.5), (2.6) and (2.7) for the p th cyclotomic field $\mathbf{Q}(\zeta_p)$, respectively.

§ 2. Results

Let m be an integer with $m \geq 3$ and $m \not\equiv 2 \pmod{4}$. For an integer u let $R_m(u)$ and $R'_m(u)$ be the integers such that

$$R_m(u) \equiv u \pmod{m}, \quad 0 \leq R_m(u) < m$$

and

$$R'_m(u) \equiv u \pmod{m}, \quad -\frac{m}{2} \leq R'_m(u) < \frac{m}{2},$$

respectively. For an integer u coprime to m , let u^{-1} be an integer such that $uu^{-1} \equiv 1 \pmod{m}$.

Let K be an imaginary abelian number field of degree $2n = [K : \mathbf{Q}]$ and with conductor m . Let h_K^* , Q_K and w_K be the relative class number of K , the unit index of K and the number of roots of unity in K , respectively.

Let G_m be the multiplicative group $(\mathbf{Z}/m\mathbf{Z})^\times$, \mathbf{Z} the ring of integers, and H the subgroup of G_m corresponding to K . For an integer t coprime to m let $\bar{t} = t + m\mathbf{Z} \in G_m$.

Since H does not contain $\bar{-1}$, we can take classes C_1, C_2, \dots, C_n of G_m/H satisfying

$$G_m/H = \{C_1, -C_1, C_2, -C_2, \dots, C_n, -C_n\}.$$

Let $\mathcal{R} = \{C_1, C_2, \dots, C_n\}$ and let $C_1 = H$.

Let X^+ and X^- be the sets of primitive even and odd Dirichlet characters of K , respectively. In the following, except to specify, we assume that the characters we consider are primitive.

Let b be an integer with $b \geq 2$ and $m \nmid b$. Let $m' = m/(m, b)$, $b' = b/(m, b)$, where (m, b) is the greatest common divisor of m and b .

For a character $\chi \in X^-$ let f_χ be the conductor of χ and define $c_\chi(b)$ as

$$c_\chi(b) = \begin{cases} b \prod_{l|m} (1 - \bar{\chi}(l)) & \text{if } f_\chi \nmid m', \\ b \prod_{l|m} (1 - \bar{\chi}(l)) - \frac{\varphi(m)}{\varphi(m')} \chi(b') \prod_{l|m'} (1 - \bar{\chi}(l)) & \text{if } f_\chi \mid m', \end{cases}$$

where l runs over prime numbers, $\bar{\chi}$ is the conjugate character of χ and φ the Euler totient function.

Let \tilde{K} be the composite of K and a quadratic field $\mathbf{Q}(\sqrt{D})$, where D is the discriminant of the field $\mathbf{Q}(\sqrt{D})$. We assume that D is coprime to m . Let f be the conductor of the field $\mathbf{Q}(\sqrt{D})$ and ψ the quadratic Dirichlet character corresponding to $\mathbf{Q}(\sqrt{D})$.

For a class $C = \bar{c}H$ in G_m/H let

$$T_{C, \psi}^{(b)} = \sum_{\bar{t} \in H} \sum_{d=0}^{f-1} \psi(R_m(ct) + dm) \left(\left[b \cdot \frac{R_m(ct) + dm}{fm} \right] - \frac{b-1}{2} \right),$$

where $[x]$ means the integral part of a rational number x . When $H = \{\bar{1}\}$, we use $T_{c,\psi}^{(b)}$ instead of $T_{C,\psi}^{(b)}$ for $C = \bar{c}H = \{\bar{c}\}$.

Let $h_{\tilde{K}}^*$, $Q_{\tilde{K}}$, $w_{\tilde{K}}$ and $\tilde{c}_\chi(b)$ be defined for \tilde{K} as above. Note that we define $\tilde{c}_\chi(b)$ by using fm instead of m .

Theorem 2.1. *Let K be an imaginary abelian number field of degree $2n$ and with conductor m . Let \tilde{K} , ψ and f be as above. Take an integer b with $b \geq 2$ and $fm \nmid b$. Then we have*

$$(2.1) \quad \det \left(T_{C_i C_j^{-1}, \psi}^{(b)} \right)_{C_i, C_j \in \mathcal{R}} = \prod_{\chi \in X^*} \tilde{c}_{\chi\psi}(b) \cdot \prod_{\chi \in X^*} \frac{1}{2} B_{1, \chi\psi} \\ = (-1)^n \prod_{\chi \in X^*} \tilde{c}_{\chi\psi}(b) \cdot \frac{Q_K w_K}{Q_{\tilde{K}} w_{\tilde{K}}} \cdot \frac{h_{\tilde{K}}^*}{h_K^*},$$

where X^* is X^+ or X^- according as $\psi(-1) = -1$ or $\psi(-1) = +1$.

When $b = fm + 1$, we have the following formula, which is obtained by taking $k = 1$ in [2, Theorem 1]:

Corollary 2.2 (cf. [2, Theorem 1]). *Let K be an imaginary abelian number field of degree $2n$ and with conductor m . Let \tilde{K} , ψ and f be as above. Then we have*

$$(2.2) \quad \det \left(\sum_{\bar{i} \in H} \sum_{d=0}^{f-1} \psi(R_m(c_i c_j^{-1} t) + dm) \left(\frac{R_m(c_i c_j^{-1} t) + dm}{fm} - \frac{1}{2} \right) \right)_{C_i, C_j \in \mathcal{R}} \\ = \prod_{\chi \in X^*} \prod_{l|m} (1 - \chi\psi(l)) \cdot \prod_{\chi \in X^*} \frac{1}{2} B_{1, \chi\psi} \\ = (-1)^n \prod_{\chi \in X^*} \prod_{l|m} (1 - \chi\psi(l)) \cdot \frac{Q_K w_K}{Q_{\tilde{K}} w_{\tilde{K}}} \cdot \frac{h_{\tilde{K}}^*}{h_K^*},$$

where $C_i = \bar{c}_i H$, $C_j = \bar{c}_j H$.

Let ψ_0 be the principal character modulo f and let

$$T_{C, \psi_0}^{(b)} = \sum_{\bar{i} \in H} \sum_{d=0}^{f-1} \psi_0(R_m(ct) + dm) \left(\left[b \cdot \frac{R_m(ct) + dm}{fm} \right] - \frac{b-1}{2} \right)$$

for $C = \bar{c}H$.

We remark that we have already obtained

$$(2.3) \quad \det \left(T_{C_i C_j^{-1}, \psi_0}^{(b)} \right)_{C_i, C_j \in \mathcal{R}} = \prod_{\chi \in X^-} \tilde{c}_\chi(b) \cdot \prod_{\chi \in X^-} \frac{1}{2} B_{1, \chi}$$

(See [12, (24)]) and

$$(2.4) \quad \det \left(T_{C_i C_j^{-1}, \psi_0}^{(b)} \right)_{C_i, C_j \in \mathcal{R}} \det \left(T_{C_i C_j^{-1}, \psi}^{(b)} \right)_{C_i, C_j \in \mathcal{R}} = \prod_{\chi \in \tilde{X}^-} \tilde{c}_\chi(b) \cdot \prod_{\chi \in \tilde{X}^-} \frac{1}{2} B_{1, \chi} \\ = \prod_{\chi \in \tilde{X}^-} \tilde{c}_\chi(b) \cdot \frac{h_{\tilde{K}}^*}{Q_{\tilde{K}} w_{\tilde{K}}},$$

where \tilde{X}^- is the set of odd characters of \tilde{K} . Kučera [16] gave a determinant formula generalizing the formula (2.4) but he did not refer to the formula (2.3).

Endô [6, Theorem] gave the formula (2.4), up to sign, in the case where $K = \mathbb{Q}(\zeta_{p^\mu})$, a p -power-th cyclotomic field, $\tilde{K} = \mathbb{Q}(\zeta_{p^\mu}, \sqrt{f})$, $(f, p) = 1$ and $b = fp^\mu + 1$.

We can not get our result (2.1) "directly" from (2.3) and (2.4), because

$$\det \left(T_{C_i C_j^{-1}, \psi_0}^{(b)} \right)_{C_i, C_j \in \mathcal{R}} \text{ is equal to zero under some conditions.}$$

In Theorem 2.1, taking $b = fm + 1$; $\psi = \chi_4$, $\psi = \chi_4 \psi_8$ and $\psi = \psi_8$, we have generalizations of Endô's formulas for the m th cyclotomic field $\mathbb{Q}(\zeta_m)$, m an odd integer, in [2, Theorem 2], where χ_4 is the odd character with conductor 4 and ψ_8 the even character with conductor 8:

Corollary 2.3 (cf. [2, Theorem 2]). *Let K be an imaginary abelian number field of degree $2n$ and with odd conductor m . Then we have*

$$(2.5) \quad \det \left(\sum_{\bar{t} \in H} (-1)^{R'_m(c_i c_j^{-1} t)} \right)_{C_i, C_j \in \mathcal{R}} \\ = \chi_4(m)^n 2^n \prod_{\chi \in X^+} \chi(2) \cdot \prod_{\chi \in X^+} \prod_{l|m} (1 - \chi \chi_4(l)) \cdot \frac{Q_K w_K}{Q_{K(\sqrt{-1})} w_{K(\sqrt{-1})}} \cdot \frac{h_{K(\sqrt{-1})}^*}{h_K^*},$$

$$(2.6) \quad \det \left(\sum_{\bar{t} \in H} T'_m(c_i c_j^{-1} t) \right)_{C_i, C_j \in \mathcal{R}} \\ = \chi_4 \psi_8(m)^n \prod_{\chi \in X^+} \chi(2) \cdot \prod_{\chi \in X^+} \prod_{l|m} (1 - \chi \chi_4 \psi_8(l)) \cdot \frac{Q_K w_K}{Q_{K(\sqrt{-2})} w_{K(\sqrt{-2})}} \cdot \frac{h_{K(\sqrt{-2})}^*}{h_K^*}$$

and

$$(2.7) \quad \det \left(\sum_{\bar{i} \in H} U'_m(c_i c_j^{-1} t) \right)_{C_i, C_j \in \mathcal{R}} \\ = (-1)^{\frac{m+1}{2}n} \psi_8(m)^n \prod_{\chi \in X^+} \chi(2) \cdot \prod_{\chi \in X^-} \prod_{l|m} (1 - \chi \psi_8(l)) \cdot \frac{Q_K w_K}{Q_{K(\sqrt{2})} w_{K(\sqrt{2})}} \cdot \frac{h_{K(\sqrt{2})}^*}{h_K^*},$$

where $C_i = \bar{c}_i H$, $C_j = \bar{c}_j H$ and for an integer c

$$T'_m(c) = \begin{cases} (-1)^{\frac{R'_m(c)}{2}} & \text{if } R'_m(c) \equiv 0 \pmod{2}, \\ 0 & \text{if } R'_m(c) \equiv 1 \pmod{2} \end{cases}$$

and

$$U'_m(c) = \begin{cases} 0 & \text{if } R'_m(c) \equiv 0 \pmod{2}, \\ (-1)^{\frac{R'_m(c)-1}{2}} & \text{if } R'_m(c) \equiv 1 \pmod{2}. \end{cases}$$

Endô's formula in [2, Theorem 2] are represented, up to sign, by the form of product of first generalized Bernoulli numbers. Endô [5] has already given such determinants in (2.5), (2.6) and (2.7) for the p th cyclotomic field $\mathbf{Q}(\zeta_p)$.

As introduced in §1, the formula (1.4) is a special case of (2.5) for the p th cyclotomic field $\mathbf{Q}(\zeta_p)$. Here we note that if $p \equiv 3 \pmod{4}$, then $\prod_{\chi \in X^+} \chi(2) = \chi_p(2)^{(p-1)\frac{p-3}{4}} = +1$, where χ_p is a Dirichlet character with conductor p and of degree $p-1$.

In Theorem 2.1 taking $K = \mathbf{Q}(\zeta_p)$ and $b = 2$, we have formulas in [4]:

Corollary 2.4 ([4, Theorem 1]). *Let K be the p th cyclotomic field $\mathbf{Q}(\zeta_p)$, p an odd prime. Let D be a square-free integer such that $(D, p) = 1$ and $D \equiv 1 \pmod{4}$. Let \tilde{K} be the composite of K and the quadratic field $\mathbf{Q}(\sqrt{D})$. Let $\psi(u)$ be the quadratic character corresponding to the field $\mathbf{Q}(\sqrt{D})$. For an integer u with $(u, p) = 1$ put*

$$S_u(\psi) = \sum_{\substack{(p|D|-1)/2 \\ k=1 \\ (k, pD)=1, k \equiv u \pmod{p}}} \psi(k).$$

Then we have: If $D > 0$, then

$$(2.8) \quad \det (S_{uv}(\psi))_{1 \leq u, v \leq (p-1)/2} = \pm \prod_{i=1}^{(p-1)/2} (2 - \chi_p^{2i-1} \psi(2)) \frac{1}{2} B_{1, \chi_p^{2i-1} \psi} \\ = \pm \frac{2p}{Q_{\tilde{K}} w_{\tilde{K}}} \prod_{i=1}^{(p-1)/2} (2 - \psi \chi_p^{2i-1}(2)) \cdot \frac{h_{\tilde{K}}^*}{h_K^*}.$$

If $D < 0$, then

$$(2.9) \det(S_{uv}(\psi))_{1 \leq u, v \leq (p-1)/2} = \pm (1 - \psi(p)) \prod_{i=1}^{(p-1)/2} (2 - \chi_p^{2i} \psi(2)) \frac{1}{2} B_{1, \chi_p^{2i} \psi} \\ = \pm (1 - \psi(p)) \frac{2p}{Q_{\tilde{K}} w_{\tilde{K}}} \prod_{i=1}^{(p-1)/2} (2 - \psi \chi_p^{2i}(2)) \cdot \frac{h_{\tilde{K}}^*}{h_K^*}.$$

In Theorem 2.1 taking $K = \mathbf{Q}(\zeta_p)$ and $b = g$, g a primitive root modulo p , we have a formula corresponding to the one in [11, Corollary 2]:

Corollary 2.5. *Let K be the p th cyclotomic field $\mathbf{Q}(\zeta_p)$, p an odd prime. Let g be a primitive root modulo p with $g \geq 2$ and $g \equiv 1 \pmod{4}$. Let \tilde{K} be the composite of K and the quadratic field $\mathbf{Q}(\sqrt{-1})$. Then for an integer i we have*

$$T_{R_p(g^i), \chi_4}^{(g)} = \sum_{\substack{0 \leq k \leq p-2 \\ R_{4p}(g^k) \equiv R_p(g^i) \pmod{p}}} \chi_4(R_{4p}(g^k)) \left[\frac{g R_{4p}(g^k)}{4p} \right] \\ + \sum_{\substack{0 \leq k \leq p-2 \\ R_{4p}(-g^k) \equiv R_p(g^i) \pmod{p}}} \chi_4(R_{4p}(-g^k)) \left[\frac{g R_{4p}(-g^k)}{4p} \right] \\ = \sum_{\substack{0 \leq k \leq p-2 \\ R_{4p}(g^k) \equiv R_p(g^i) \pmod{p}}} \left[\frac{g R_{4p}(g^k)}{4p} \right] - \sum_{\substack{0 \leq k \leq p-2 \\ R_{4p}(-g^k) \equiv R_p(g^i) \pmod{p}}} \left[\frac{g R_{4p}(-g^k)}{4p} \right]$$

and

$$\det \left(T_{R_p(g^{i-j}), \chi_4}^{(g)} \right)_{0 \leq i, j \leq (p-3)/2} = (-1)^{\frac{p-1}{2}} \frac{1}{4} (1 - \chi_4(p)) \prod_{\chi \in X^+} (g - \chi(g)) \cdot \frac{h_{\tilde{K}}^*}{h_K^*}.$$

Remark. With the notation in Corollary 2.5 expand $1/(4p)$ to the basis $1/g$:

$$\frac{1}{4p} = \sum_{k=1}^{\infty} \frac{x(k)}{g^k}, \quad x(k) \in \{0, 1, \dots, g-1\}.$$

Then we have

$$x(k) = \left[\frac{g R_{4p}(g^{k-1})}{4p} \right] \quad \text{for } k = 1, 2, \dots$$

(See [11, Theorem 10]).

Example 2.6. We give here an example of Corollary 2.5. Let K , \tilde{K} and g be

as in Corollary 2.5. We take $p = 7$, $g = 5$. Then $K = \mathbf{Q}(\zeta_7)$, $\tilde{K} = \mathbf{Q}(\zeta_{28})$ and

$$\begin{aligned} T_{R_7(5^0), \chi_4}^{(5)} &= \left[\frac{5 \cdot 1}{28} \right] - \left[\frac{5 \cdot 15}{28} \right] = 0 - 2 = -2, \\ T_{R_7(5^1), \chi_4}^{(5)} &= \left[\frac{5 \cdot 5}{28} \right] - \left[\frac{5 \cdot 19}{28} \right] = 0 - 3 = -3, \\ T_{R_7(5^2), \chi_4}^{(5)} &= \left[\frac{5 \cdot 25}{28} \right] - \left[\frac{5 \cdot 11}{28} \right] = 4 - 1 = 3, \\ T_{R_7(5^3), \chi_4}^{(5)} &= \left[\frac{5 \cdot 13}{28} \right] - \left[\frac{5 \cdot 27}{28} \right] = 2 - 4 = -2, \\ T_{R_7(5^4), \chi_4}^{(5)} &= \left[\frac{5 \cdot 9}{28} \right] - \left[\frac{5 \cdot 23}{28} \right] = 1 - 4 = -3, \\ T_{R_7(5^5), \chi_4}^{(5)} &= \left[\frac{5 \cdot 17}{28} \right] - \left[\frac{5 \cdot 3}{28} \right] = 3 - 0 = 3. \end{aligned}$$

Hence

$$\det \left(T_{R_7(5^{i-j}), \chi_4}^{(5)} \right)_{0 \leq i, j \leq 2} = \det \begin{pmatrix} -2 & 3 & -3 \\ -3 & -2 & 3 \\ 3 & -3 & -2 \end{pmatrix} = -62.$$

On the other hand, letting ζ_u be a primitive u th root of unity, we have

$$\begin{aligned} (-1)^{\frac{7-1}{2}} \frac{1}{4} (1 - \chi_4(7)) \prod_{\chi \in X^+} (5 - \chi(5)) \cdot \frac{h_{\tilde{K}}^*}{h_K^*} &= -\frac{1}{4} \cdot 2 \cdot (5-1)(5-\zeta_3)(5-\zeta_3^2) \cdot \frac{h_{\tilde{K}}^*}{h_K^*} \\ &= -62 \cdot \frac{h_{\tilde{K}}^*}{h_K^*}. \end{aligned}$$

Therefore $h_{\tilde{K}}^* = h_K^*$. Actually, we have already known that $h_{\tilde{K}}^* = h_K^* = 1$.

§ 3. Proofs of Theorem and Corollaries

To prove Theorem 2.1 we need the following lemma originating from [8].

Lemma 3.1 ([14, Lemma 1]). *Let K be an imaginary abelian number field with conductor m and b an integer with $b \geq 2$ and $m \nmid b$. Then, for an odd character χ of K and for $C = \bar{c}H$ we have*

$$\begin{aligned} \frac{1}{2} c_\chi(b) B_{1, \bar{\chi}} &= \sum_{C \in \mathcal{R}} \bar{\chi}(c) T_C^{(b)} \\ &= \frac{1}{2} \sum_{\substack{k=1 \\ (k, m)=1}}^m \bar{\chi}(k) \left(\left[b \cdot \frac{R_m(k)}{m} \right] - \frac{b-1}{2} \right), \end{aligned}$$

where

$$T_C^{(b)} = \sum_{\bar{t} \in H} \left(\left[b \cdot \frac{R_m(ct)}{m} \right] - \frac{b-1}{2} \right).$$

Proof of Theorem 2.1. Recall that for a class $C = \bar{c}H$ in G_m/H we define

$$T_{C,\psi}^{(b)} = \sum_{\bar{t} \in H} \sum_{d=0}^{f-1} \psi(R_m(ct) + dm) \left(\left[b \cdot \frac{R_m(ct) + dm}{fm} \right] - \frac{b-1}{2} \right).$$

Let $\psi(-1) = (-1)^{k'}$, $k' = 0$ or 1 . Then we have

$$T_{-C,\psi}^{(b)} = (-1)^{k'+1} T_{C,\psi}^{(b)}.$$

First we consider the case where $\psi(-1) = -1$. Since $T_{-C,\psi}^{(b)} = T_{C,\psi}^{(b)}$, it follows from the group determinant (cf. for example, [19, p.71]) that

$$\det(T_{C_i C_j^{-1}, \psi}^{(b)})_{C_i, C_j \in \mathcal{R}} = \prod_{\chi \in X^+} \sum_{\bar{c}H \in \mathcal{R}} \chi(c) T_{\bar{c}H, \psi}^{(b)}.$$

Since

$$\sum_{\bar{c}H \in \mathcal{R}} \chi(c) T_{\bar{c}H, \psi}^{(b)} = \frac{1}{2} \sum_{\substack{k=1 \\ (k, fm)=1}}^{fm} \chi\psi(k) \left(\left[\frac{bk}{fm} \right] - \frac{b-1}{2} \right),$$

noting that the assumption of b , we obtain by Lemma 3.1

$$\sum_{\bar{c}H \in \mathcal{R}} \chi(c) T_{\bar{c}H, \psi}^{(b)} = \frac{1}{2} \tilde{c}_{\chi\psi}(b) B_{1, \chi\psi}.$$

Therefore we have

$$\begin{aligned} \det(T_{C_i C_j^{-1}, \psi}^{(b)})_{C_i, C_j \in \mathcal{R}} &= \prod_{\chi \in X^+} \tilde{c}_{\chi\psi}(b) \cdot \prod_{\chi \in X^+} \frac{1}{2} B_{1, \chi\psi} \\ &= (-1)^n \prod_{\chi \in X^*} \tilde{c}_{\chi\psi}(b) \cdot \frac{Q_K w_K}{Q_{\tilde{K}} w_{\tilde{K}}} \cdot \frac{h_{\tilde{K}}^*}{h_K^*}. \end{aligned}$$

Next we consider the case where $\psi(-1) = +1$. Taking some odd character χ_1 of

K , we have, letting $C_i = \bar{c}_i H$, $C_j = \bar{c}_j H$,

$$\begin{aligned}
 \det(T_{C_i C_j^{-1}, \psi}^{(b)})_{C_i, C_j \in \mathcal{R}} &= \det(\chi_1(c_i c_j^{-1}) T_{C_i C_j^{-1}, \psi}^{(b)})_{C_i, C_j \in \mathcal{R}} \\
 &= \prod_{\chi \in X^+} \sum_{\bar{a} H \in \mathcal{R}} \chi_1 \chi(a) T_{\bar{a} H, \psi}^{(b)} \\
 &= \prod_{\chi \in X^-} \sum_{\bar{a} H \in \mathcal{R}} \chi(a) T_{\bar{a} H, \psi}^{(b)} \\
 &= \prod_{\chi \in X^-} \tilde{c}_{\chi \psi}(b) \cdot \prod_{\chi \in X^-} \frac{1}{2} B_{1, \chi \psi} \\
 &= (-1)^n \prod_{\chi \in X^*} \tilde{c}_{\chi \psi}(b) \cdot \frac{Q_K w_K}{Q_{\tilde{K}} w_{\tilde{K}}} \cdot \frac{h_{\tilde{K}}^*}{h_K^*}.
 \end{aligned}$$

This completes the proof. \square

Corollary 2.2 is easily proved by Theorem 2.1, because when $b = fm + 1$, it holds that

$$\tilde{c}_{\chi \psi}(b) = fm \prod_{l|fm} (1 - \bar{\chi} \psi(l)) = fm \prod_{l|m} (1 - \bar{\chi} \psi(l)).$$

To prove Corollary 2.3 we need the following two lemmas.

Lemma 3.2 (cf. [13, Lemma 2]). *For an integer c with $(c, m) = 1$, we define the permutation σ_c on $\mathcal{R} = \{C_1, C_2, \dots, C_n\}$ up to “ \pm ” by*

$$\bar{c} C_i = \pm C_{\sigma_c(i)} \quad \text{for } i = 1, 2, \dots, n.$$

Then we have

$$(3.1) \quad \text{sgn } \sigma_c = \prod_{\chi \in X^+} \chi(c),$$

where $\text{sgn } \sigma_c = +1$ or -1 according as σ_c is even or odd.

In [13, Lemma 2] we have shown that

$$(-1)^{N_c} \text{sgn } \sigma_c = \prod_{\chi \in X^-} \chi(c),$$

where N_c is the number of the “minus cosets” $-C_{\sigma_c(i)}$ in the set $\{\bar{c} C_i = \pm C_{\sigma_c(i)}; i = 1, 2, \dots, n\}$. We can prove the identity (3.1) by taking $\prod_{\chi \in X^+} \chi(c)$ instead of $\prod_{\chi \in X^-} \chi(c)$ in the proof of [13, Lemma 2]. (The right hand $\delta_{ij} \zeta_{g_i}$ of the equation on page 22, line 13 from the top of [13] should be $\zeta_{g_i}^{\delta_{ij}}$.)

Lemma 3.3 ([2, Proof of Theorem 2]). Assume that m is an odd integer. For an integer c with $(c, m) = 1$ let c' an integer such that $(c', m) = 1$ and $2c' \equiv c \pmod{m}$. Then, for any integers c_i, c_j coprime to m , we have

$$\sum_{d=0}^3 \chi_4(R_m(c_i c_j'^{-1}) + dm) \left(\frac{R_m(c_i c_j'^{-1}) + dm}{4m} - \frac{1}{2} \right) = -\chi_4(m) \cdot \frac{1}{2} \cdot (-1)^{R'_m(c_i c_j'^{-1})},$$

$$\sum_{d=0}^7 \chi_4 \psi_8(R_m(c_i c_j'^{-1}) + dm) \left(\frac{R_m(c_i c_j'^{-1}) + dm}{8m} - \frac{1}{2} \right) = -\chi_4 \psi_8(m) T'_m(c_i c_j'^{-1})$$

and

$$\sum_{d=0}^7 \psi_8(R_m(c_i c_j'^{-1}) + dm) \left(\frac{R_m(c_i c_j'^{-1}) + dm}{8m} - \frac{1}{2} \right) = (-1)^{\frac{m-1}{2}} \psi_8(m) U'_m(c_i c_j'^{-1}).$$

Proof of Corollary 2.3. By Lemmas 3.3 and 3.2 and by Corollary 2.2 we have

$$\begin{aligned} & (-1)^n \chi_4(m)^n \cdot \frac{1}{2^n} \cdot \det \left(\sum_{\bar{t} \in H} (-1)^{R'_m(c_i c_j^{-1} \bar{t})} \right)_{C_i, C_j \in \mathcal{R}} \\ &= \det \left(\sum_{\bar{t} \in H} \sum_{d=0}^3 \chi_4(R_m(c_i c_j'^{-1} \bar{t}) + dm) \left(\frac{R_m(c_i c_j'^{-1} \bar{t}) + dm}{4m} - \frac{1}{2} \right) \right)_{C_i, C_j \in \mathcal{R}} \\ &= \prod_{\chi \in X^+} \chi(2) \cdot \det \left(\sum_{\bar{t} \in H} \sum_{d=0}^3 \chi_4(R_m(c_i c_j'^{-1} \bar{t}) + dm) \left(\frac{R_m(c_i c_j'^{-1} \bar{t}) + dm}{4m} - \frac{1}{2} \right) \right)_{C_i, C_j \in \mathcal{R}} \\ &= \prod_{\chi \in X^+} \chi(2) \cdot (-1)^n \prod_{\chi \in X^+} \prod_{l|m} (1 - \chi \chi_4(l)) \cdot \frac{Q_K w_K}{Q_{K(\sqrt{-1})} w_{K(\sqrt{-1})}} \cdot \frac{h_K^*(\sqrt{-1})}{h_K^*}, \end{aligned}$$

where $C_i = \bar{c}_i H$, $C_j = \bar{c}_j H$. Hence we have obtained the first formula (2.5).

In the same way as above we can prove the second and third formulas (2.6) and (2.7). \square

Proof of Corollary 2.4. If $\psi(-1) = -1$, then we have $T_{-C, \psi}^{(b)} = -T_{C, \psi}^{(b)}$ and

$$\det \left(T_{C_i C_j^{-1}, \psi}^{(b)} \right)_{C_i, C_j \in \mathcal{R}} = (-1)^{\frac{n-\delta}{2} + \delta'} \det \left(T_{C_i C_j, \psi}^{(b)} \right)_{C_i, C_j \in \mathcal{R}}.$$

If $\psi(-1) = +1$, then we have $T_{-C, \psi}^{(b)} = T_{C, \psi}^{(b)}$ and

$$\det \left(T_{C_i C_j^{-1}, \psi}^{(b)} \right)_{C_i, C_j \in \mathcal{R}} = (-1)^{\frac{n-\delta}{2}} \det \left(T_{C_i C_j, \psi}^{(b)} \right)_{C_i, C_j \in \mathcal{R}}.$$

Here δ is the number of the cosets $C_i \in \mathcal{R}$ whose square C_i^2 are H or $-H$, and δ' the number of cosets $C_i \in \mathcal{R}$ whose inverses are not contained in \mathcal{R} , i.e., $C_i^{-1} = -C_{\sigma(i)}$ for some $\sigma(i) \in \{1, 2, \dots, n\}$ (As for the signs of the two identities of determinants just above, see [9, Proposition 2]).

Therefore by Theorem 2.1 we obtain, in the both cases where $\psi(-1) = \pm 1$,

$$\begin{aligned} \det \left(T_{C_i C_j, \psi}^{(b)} \right)_{C_i, C_j \in \mathcal{R}} &= \pm \prod_{\chi \in X^*} \tilde{c}_{\chi\psi}(b) \cdot \prod_{\chi \in X^*} \frac{1}{2} B_{1, \chi\psi} \\ &= \pm \prod_{\chi \in X^*} \tilde{c}_{\chi\psi}(b) \cdot \frac{Q_K w_K}{Q_{\tilde{K}} w_{\tilde{K}}} \cdot \frac{h_{\tilde{K}}^*}{h_K^*}. \end{aligned}$$

In our case where $K = \mathbb{Q}(\zeta_p)$, $H = \{\bar{1}\}$ and $b = 2$, we have $T_{C, \psi}^{(2)} = -S_c(\psi)$ for $C = \bar{c}H$. Hence, calculating $\tilde{c}_{\chi\psi}(2)$ we have the desired formula. \square

Corollary 2.5 immediately follows from Theorem 2.1 by taking $b = g$. Here we only note that $Q_K = 1$ and $Q_{\tilde{K}} = 2$ and that the multiplicative group $(\mathbb{Z}/4p\mathbb{Z})^\times$ constitutes of $R_{4p}(g^k)$ modulo $4p$ and $R_{4p}(-g^k)$ modulo $4p$ for all $k = 0, 1, \dots, p-2$.

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